Yin, Zhi; **Marciniak, Marcin**; **Horodecki, Michał**

Operator space approach to steering inequality. (English) Zbl 1342.81059

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Before we are going to start with a more detailed review of this paper, dealing with a complex and highly fascinating topic at the interface on foundations of quantum mechanics, operator theory and operator algebras, philosophy and logics, we firstly shed somewhat light on the background of the underlying main subject to get a better understanding of the content of the paper under review.

One of the most controversial discussions in quantum theory is the one of so called "hidden variables"; that is, the question of whether in principle it is possible to describe quantum statistics by means of classical probability theory in the sense of A. Kolmogorov. The first attempt to prove the impossibility of the existence of hidden variables in quantum theory was made by J. von Neumann. In 1966, J. Bell, by analysing the Einstein-Podolski-Rosen paradox, showed the incompleteness of von Neumann's "proof". Von Neumann's argumentation builds on the assumption that physical reality is "non-contextual" (i.e., independent of the measurement arrangement). Bell analysed a further fundamental property of the quantum-mechanical description, known as "nonseparability" (of states). Mathematically it is related to the superposition principle and to the fact that combined quantum systems are described by tensor products rather than by cartesian products of classical Kolmogorovian probability theory.

Many of these still ongoing discussions concentrate on the implications of the accepted fact that under the assumption that any impact of a subsystem towards a spatially distant subsystem cannot be transmitted faster than light (the so called "Principle of Locality") the von Neumann model of quantum mechanics is an incomplete theory which cannot be completed due to the introduction of "hidden variables". If such a completion was possible the assumed Principle of Locality would then imply that the von Neumann model of quantum mechanics can be extended to a framework where one could allocate concretely defined values to all observables – independent of the process of measurement, implying that they would be jointly measurable on a *single* classical Kolmogorovian probability space on which "hidden variables" emerge as related random variables, implying a so called "local realism". But then all observables (represented by possibly unbounded linear operators on a Hilbert space in the von Neumann model) already would commute which obviously would lead to a contradiction.

Moreover, it is also experimentally verified that such entangled composite quantum systems violate certain relations between correlations – known as "Bell's inequalities". Purely in terms of of a very elementary application of classical Kolmogorovian probability theory – and completely independent of any modelling assumptions in physics – Bell's inequalities can be represented in form of an inequality originating from [*J. F. Clauser* et al., Rev. Lett. 23, No. 15, 880–883 (1969; Zbl 1371.81014); erratum ibid. 24, No. 10, 549 (1970)]:

BCHSH Inequality. Let $(\Omega, \mathcal{F})$ be an arbitrary measurable space. Let X_1, X_2, X_3 and X_4 be arbitrary bounded random variables with values in $[-1, 1]$, all defined on $(\Omega, \mathcal{F})$. Then

$$|\mathbb{E}_{\mathbb{P}}[X_1X_2] - \mathbb{E}_{\mathbb{P}}[X_1X_3]| \leq 1 - \mathbb{E}_{\mathbb{P}}[X_2X_3]$$

for all probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F})$. In particular,

$$|\mathbb{E}_{\mathbb{P}}[X_1X_2] - \mathbb{E}_{\mathbb{P}}[X_1X_3] + \mathbb{E}_{\mathbb{P}}[X_4X_2] + \mathbb{E}_{\mathbb{P}}[X_4X_3]| \leq 2\,,$$

for all probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F})$.

To see one significant implication of the BCHSH inequality, assume that there are three measurable spaces $(\Omega_i, \mathcal{F}_i)$ $(i = 1, 2, 3)$ and three bounded random variables $X_i : \Omega_i \longrightarrow [-1, 1]$. Let us consider three product σ-algebras $\mathcal{G}_1 := \mathcal{F}_2 \otimes \mathcal{F}_3$, $\mathcal{G}_2 := \mathcal{F}_1 \otimes \mathcal{F}_3$ and $\mathcal{G}_3 := \mathcal{F}_1 \otimes \mathcal{F}_2$, and let us further assume the existence of three probability measures $\mathbb{P}_i$, defined on $\mathcal{G}_i$ $(i = 1, 2, 3)$ such that

$$|\mathbb{E}_{\mathbb{P}_3}[X_1X_2] - \mathbb{E}_{\mathbb{P}_2}[X_1X_3]| > 1 - \mathbb{E}_{\mathbb{P}_1}[X_2\,X_3]\,.$$

Then the conclusion would be that there cannot exist a "superior" probability measure $\mathbb{P}$ on the measurable space $(\Omega_1 \times \Omega_2 \times \Omega_3, \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3)$ such that for all $i \in \{1, 2, 3\}$ and for any $A \in \mathcal{G}_i$ we have $\mathbb{P}(A \times \Omega_i) = \mathbb{P}_i$. In other words, there cannot exist a joint distribution function of the random vector (X_1, X_2, X_3) whose bivariate marginals are given by the three $\mathbb{P}_i$ s! This fact reflects a link between BCHSH inequalities and the question whether there exist joint distributions with given multivariate marginals, known as the "Marginal Problem" – which already is revealed in classical commutative probability theory in the sense of Kolmogorov (cf. Theorem 1.27 in [*L. Rüschendorf*, Mathematical risk analysis. Dependence, risk bounds, optimal allocations and portfolios. Berlin: Springer (2013; Zbl 1266.91001)]).

As it is well-known from quantum mechanics and quantum information such a violation of the BCHSH inequality can be explicitly realised in form of laboratory experiments (induced by entangled pure spin states in composite quantum systems), implying that in general a probabilistic interpretation of quantum states in the sense of Kolmogorov is not well-defined due to the existence of incompatible observables (i.e., physical objects which cannot be measured simultaneously). In particular, due to the existence of incompatible (non-commuting) observables there is no direct analogue of conditional probability in the sense of Kolmogorov (since there is no commutative "AND" conjunction of these observables). Moreover, as sketched above there does not exist a "superior" multivariate distribution function, implying that one cannot look for a single probability measure in the sense of Kolmogorov to calculate the probability of a simultaneous measurement of two incompatible observables.

We should also mention an important characterisation of arbitrary quantum probability measures, known as Gleason's Theorem (cf. [*A. M. Gleason*, J. Math. Mech. 6, 885–893 (1957; Zbl 0078.28803)] and [*V. Moretti*, Spectral theory and quantum mechanics. With an introduction to the algebraic formulation. Translated by Simon G. Chiossi. Translated and extended edition of the 2010 Italian original. Milano: Springer (2013; Zbl 1365.81001)]):

Gleason's Theorem. Let H be a complex Hilbert space which is either of finite dimension ≥ 3 or infinite-dimensional and separable. If $T \in D \in \mathcal{N}(H)$ is a positive (self-adjoint) nuclear operator such that $\mathbf{N}(D) = 1 = \text{tr}(D)$ then $P \mapsto tr(TP)$ defines a quantum probability measure on the set of all orthogonal projectons on H. Conversely, for any quantum probability measure μ there exists a unique positive nuclear operator $D \in \mathcal{N}(H)$ such that $\mathbf{N}(D) = 1 = \text{tr}(D)$ and

$$\mu(P) = \text{tr}(PD)$$

for all orthogonal projectons in the class $\mathcal{L}(H)$ of all bounded linear operators on H.

The normalised nuclear positive operator D is known as "density matrix" (which could be viewed as the non-commutative generalisation of the Radon-Nikodym derivative in classical measure theory). To understand the latter notation let us recall that from Banach space theory we know that every normal state (i.e., every weak-* continuous positive linear form of norm one) ϕ on the von Neumann algebra $\mathcal{L}(H) \cong (H^* \widetilde{\otimes}_\pi H)^* \cong \mathcal{N}(H)^*$ in fact is completely characterised by $\phi = \text{tr}(\cdot\, D)$ for a unique nuclear positive operator D, satisfying $\text{tr}(D) = 1 = \mathbf{N}(D)$ (cf. e.g. [*H. Araki*, Mathematical theory of quantum fields. Transl. from the Japanese by Ursula Carow-Watamura. Oxford: Oxford University Press (1999; Zbl 0998.81501)] and [*B. Blackadar*, Operator algebras. Theory of C^*-algebras and von Neumann algebras. Berlin: Springer (2006; Zbl 1092.46003)]).

Here, one should be aware that not only the interpretation of quantum mechanics but also the presentation of the mathematical formalism shows a somewhat confusing variation in the comprehensive literature including the construction of analogies between objects from quantum probability theory and objects from Kolmogorovian probability theory. For example, one can find descriptions of the mathematical formalism of quantum mechanics in form of wave functions, projection valued measures (PVMs), positive operator valued measures (POVMs), projection lattices and von Neumann algebras including non-trivial relations between these model constructions and their physical interpretation; particularly with a view towards the fields of quantum information and algebraic quantum field theory.

In any case, since any Kolmogorovian probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in particular is a σ-finite regular measure space, it is completely encoded in the commutative von Neumann algebra $L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \stackrel{1}{\hookrightarrow} \mathcal{L}\big(L^2(\Omega, \mathcal{F}, \mathbb{P})\big)$, where a normal state corresponds to the Radon-Nikodym derivative of a probability measure which is absolutely continuous with respect to the given probability measure $\mathbb{P}$. Hence, one generalised probability space model (which includes the commutative Kolmogorovian model) is then given by $(\mathcal{A}, \phi)$, consisting of a von Neumann algebra $\mathcal{A}$ and a normal state $\phi = \text{tr}(\cdot\, D_\phi)$ on $\mathcal{A}$ (cf. e.g. [*M. Rédei* and *S. J. Summers*, Stud. Hist. Philos. Sci., Part B, Stud. Hist. Philos. Mod. Phys. 38, No. 2, 390–417 (2007; Zbl 1223.46058)]).

The BCHSH inequality, transferred to this generalised probability space model primarily reads as follows:

Generalised BCHSH Inequality. Let $\mathcal{A} \overset{1}{\hookrightarrow} \mathcal{L}(H)$ and $\mathcal{B} \overset{1}{\hookrightarrow} \mathcal{L}(H)$ be two C*-algebras. Let ϕ be an arbitrary state on $\mathcal{L}(H)$. If $\mathcal{A}$ or $\mathcal{B}$ is commutative then

$$|\phi(A(B - B'))| \leq 1 - \phi(BB')$$

for all $A \in \mathcal{A}$, $B, B' \in \mathcal{B}$, satisfying $-1_{\mathcal{A}} \leq A \leq 1_{\mathcal{A}}$ and $-1_{\mathcal{B}} \leq B, B' \leq 1_{\mathcal{B}}$. Moreover,

$$|\phi(A(B - B')) + \phi(A'(B + B'))| \leq 2 \,.$$

for all $A, A' \in \mathcal{A}$, $B, B' \in \mathcal{B}$, satisfying $-1_{\mathcal{A}} \leq A, A' \leq 1_{\mathcal{A}}$ and $-1_{\mathcal{B}} \leq B, B' \leq 1_{\mathcal{B}}$.

To complete the bridge to the paper under review let us quickly recall the fundamental notion of an entangled state. In general, composite quantum systems are modelled as completed minimal (or spatial, respectively injective operator space) tensor products of von Neumann algebras, making them again a von Neumann algebra. A normal state on such a tensor product (or equivalently, a density matrix) is separable (or decomposable) if it is a limit point of the convex hull of product normal states. Otherwise it is called entangled state (cf. also [*A. W. Majewski*, Open Syst. Inf. Dyn. 6, No. 1, 79–86 (1999; Zbl 0932.46069)]). Pure (and hence normal) states on tensor products of von Neumann algebras are separable if and only if their density matrix can be written as a canonical tensor product of all "marginal" density matrices, where each of these marginal density matrices corresponds to a pure state on the respective component of the tensor product of the von Neumann algebra. Here, it is worth to recall the following version of a crucial result reflecting the impact of non-commutativity in composite quantum systems (cf. [*B. Blackadar*, Operator algebras. Theory of C^*-algebras and von Neumann algebras. Berlin: Springer (2006; Zbl 1092.46003)], [*N. P. Landsman*, Stud. Hist. Philos. Sci., Part B, Stud. Hist. Philos. Mod. Phys. 37, No. 1, 212–242 (2006; Zbl 1222.81072)] and [*G. A. Raggio*, Lett. Math. Phys. 15, No. 1, 27–29 (1988; Zbl 0659.46061)]):

Theorem. Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras. Then the following statements on the minimal tensor product $\mathcal{A} \otimes_{\min} \mathcal{B}$ are equivalent:

(i) Each normal state on $\mathcal{A} \otimes_{\min} \mathcal{B}$ is separable.

(ii) $\mathcal{A}$ or $\mathcal{B}$ is commutative.

(iii) The state space of $\mathcal{A}$ or $\mathcal{B}$ is a simplex.

(iv) The positive elements in $\mathcal{A}$ or $\mathcal{B}$ constitute a lattice.

(v) Each normal state on $\mathcal{A} \otimes_{\min} \mathcal{B}$ satisfies the generalised BCHSH Inequality.

Consequently, the existence of normal entangled states on $\mathcal{A} \otimes_{\min} \mathcal{B}$ necessarily implies that both von Neumann algebras, $\mathcal{A}$ and $\mathcal{B}$ have to be non-commutative. However, there are entangled states for which the BCHSH inequalities still hold.

The authors of the paper under review contribute to the highly vibrating investigation of the different types of correlation in quantum mechanics (and quantum information). By analysing thoroughly properties of a class of Bell violating (and hence entangled) states which allow a so called quantum steering (cf. [*E. Schrödinger*, Proc. Camb. Philos. Soc. 31, 555–563 (1935; Zbl 0012.42702); ibid. 32, 446–452 (1936; Zbl 0015.04403)] and [*D. Cavalcanti* and *P. Skrzypczyk*, "Quantum steering: a short review with focus on semidefinite programming", Preprint, `arXiv:1604.00501`]) they exceed classical dependence modelling in Kolmogorovian probability theory, too. Given a bipartite scenario, quantum steering refers to the fact that one of the two measuring parties (Alice and Bob, say) already can change the state of the other just by applying local measurements. In [*H. M. Wiseman* et al., Phys. Rev. Lett. 98, No. 14, Article ID 140402, 4 p. (2007; Zbl 1228.81078)] quantum steering is formally described in terms of an incompatibility of quantum mechanical predictions with a locally hidden state (LHS) model, where pre-determined (i.e., locally hidden) states are sent to the parties. Furthermore, quantum steering can also be seen as entanglement detection. In general, it is well-known that for mixed states, Bell violation is strictly stronger than steering which, in turn, is strictly stronger than entanglement.

A deviation from a LHS description is quantified by a violation of the quantum steering inequality for steering functionals – similarly to the violation of the BCHSH inequalities, where the latter quantifies the deviation from the commutative Kolmogorovian LHV case (a detailed introduction to the subject of steering inequalities and steering functionals is described in the companion paper [the authors et al., "Unbounded violation of quantum steering inequalities", Phys. Rev. Lett. 115, No. 17, Article ID 170401, 5 p. (2015; doi:10.1103/PhysRevLett.115.170401)]). In general, it is rather complicated to compute the violation for a given steering functional analytically. One usually uses semi-definite programming (cf. e.g. [*D. Cavalcanti* and *P. Skrzypczyk*, "Quantum steering: a short review with focus on semidefinite programming", arXiv:1604.00501]). Instead of applying the latter method the authors apply deep methods from the theory of operator spaces including tensor products of Banach and operator spaces, allowing them to construct a sequence of steering functionals which is of unbounded largest violation in the following sense: for any $n \in \mathbb{N}$ there is a steering functional F_n (which is identified with an element in a suitable tensor product encoding the quantum mechanical nature of the steering) such that the largest quantum violation of steering inequality for F_n exceeds the number $\mathrm{K}\,\frac{\sqrt{n}}{\sqrt{\ln(n)}}$, where the constant $K > 0$ does not depend on n. A concrete example of such a sequence (F_n) of unbounded largest violation is constructed; yet with large probability (in the sense of Kolmogorov) only – confirmed explicitly by the authors in Remark 2.13. To this end, the authors transfer the operator space approach of [*M. Junge* et al., Commun. Math. Phys. 300, No. 3, 715–739 (2010; Zbl 1211.46068); ibid. 306, No. 3, 695–746 (2011; Zbl 1230.81011)] to their investigation of the violation of steering inequalities. In the latter two papers operator space theory is used to analyse a violation of the BCHSH inequalities.

Reviewer: Frank Oertel (London)

MSC:

81P40 Quantum coherence, entanglement, quantum correlations
81P15 Quantum measurement theory
81P05 General and philosophical topics in quantum theory
46L30 States of C^*-algebras
46L06 Tensor products of C^*-algebras
46L60 Applications of selfadjoint operator algebras to physics

Keywords:

steering inequality; unbounded largest violation; operator space

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